On the augmentation quotients of the $mathrm{IA}$-automorphism group of a free group (Topology of transformation groups and its related topics)

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数理解析研究所講究録
第 1876 巻 2014 年 9-15

On the augmentation quotients of the IA-automorphism group of a free group

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Abstract

In this article, we study the augmentation quotients of the IA-automorphism group of a free group and a free metabelian group. First, for any group G, we construct a lift of the k-th Johnson homomorphism of the automorphism group of G to the k-th augmentation quotient of the IA-automorphism group of G. Then we study the images of these homomorphisms for the case where G is a free group and a free metabelian group. As a corollary, we detect a $\mathbf{Z}$-free part in each of the augmentation quotients, which can not be detected by the abelianization of the IA-automorphism group. For details, see our paper [25].

Let F_n be a free group of rank $n \geq 2$, and $\mathrm{Aut}\, F_n$ the automorphism group of F_n. Let $\rho : \mathrm{Aut}\, F_n \to \mathrm{Aut}\, H$ denote the natural homomorphism induced from the abelianization $F_n \to H$. The kernel of ρ is called the IA-automorphism group of F_n, denoted by IA_n. The subgroup IA_n reflects much of the richness and complexity of the structure of $\mathrm{Aut}\, F_n$, and plays important roles in various studies of $\mathrm{Aut}\, F_n$. Although the study of the IA-automorphism group has a long history since its finitely many generators were obtained by Magnus [14] in 1935, the combinatorial group structure of IA_n is still quite complicated. For instance, no presentation for IA_n is known in general.

We have studied IA_n mainly using the Johnson filtration of $\mathrm{Aut}\, F_n$ so far. The Johnson filtration is one of a descending central series

$$\mathrm{IA}_n = \mathcal{A}_n(1) \supset \mathcal{A}_n(2) \supset \cdots$$

consisting of normal subgroups of $\mathrm{Aut}\, F_n$, whose first term is IA_n. Each graded quotient $\mathrm{gr}^k(\mathcal{A}_n) := \mathcal{A}_n(k)/\mathcal{A}_n(k+1)$ naturally has a $\mathrm{GL}(n, \mathbf{Z})$-module structure, and from it we can extract some valuable information

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about IA_n. For example, $\mathrm{gr}^1(\mathcal{A}_n)$ is just the abelianization of IA_n due to Cohen-Pakianathan [6, 7], Farb [9] and Kawazumi [13]. Pettet [19] determined the image of the cup product $\cup_{\mathbf{Q}} : \Lambda^2 H^1(\mathrm{IA}_n, \mathbf{Q}) \to H^2(\mathrm{IA}_n, \mathbf{Q})$ by using the $\mathrm{GL}(n, \mathbf{Q})$-module structure of $\mathrm{gr}^2(\mathcal{A}_n) \otimes_{\mathbf{Z}} \mathbf{Q}$. At the present stage, however, the structures of the graded quotients $\mathrm{gr}^k(\mathcal{A}_n)$ are far from well-known.

On the other hand, compared with the Johnson filtration, the lower central series $\Gamma_{\mathrm{IA}_n}(k)$ of IA_n and its graded quotients

$$\mathcal{L}_{\mathrm{IA}_n}(k) := \Gamma_{\mathrm{IA}_n}(k)/\Gamma_{\mathrm{IA}_n}(k+1)$$

are somewhat easier to handle since we can obtain finitely many generators of $\mathcal{L}_{\mathrm{IA}_n}(k)$ using the Magnus generators of IA_n. Since the Johnson filtration is central, $\Gamma_{\mathrm{IA}_n}(k) \subset \mathcal{A}_n(k)$ for any $k \geq 1$. It is conjectured that $\Gamma_{\mathrm{IA}_n}(k) = \mathcal{A}_n(k)$ for each $k \geq 1$ by Andreadakis who showed $\Gamma_{\mathrm{IA}_2}(k) = \mathcal{A}_2(k)$ for each $k \geq 1$. It is currently known that $\Gamma_{\mathrm{IA}_n}(2) = \mathcal{A}_n(2)$ due to Bachmuth [2], and that $\Gamma_{\mathrm{IA}_n}(3)$ has at most finite index in $\mathcal{A}_n(3)$ due to Pettet [19].

In this article, we consider the augmentation quotients of IA_n. Let $\mathbf{Z}[G]$ be the integral group ring of a group G, and $\Delta(G)$ the augmentation ideal of $\mathbf{Z}[G]$. We denote by $Q^k(G) := \Delta^k(G)/\Delta^{k+1}(G)$ the k-th augmentation quotient of G. The augmentation quotients $Q^k(\mathrm{IA}_n)$ of IA_n seem to be closely related to the lower central series $\Gamma_{\mathrm{IA}_n}(k)$ as follows. If the Andreadakis's conjecture is true, then each of the graded quotients $\mathcal{L}_{\mathrm{IA}_n}(k)$ is free abelian. Using a work of Sandling and Tahara [21], we obtain a conjecture for the $\mathbf{Z}$-module structure of $Q^k(\mathrm{IA}_n)$:

Conjecture 1. *For any $k \geq 1$,*

$$Q^k(\mathrm{IA}_n) \cong \sum \bigotimes_{i=1}^{k} S^{a_i}(\mathcal{L}_{\mathrm{IA}_n}(i))$$

as a $\mathbf{Z}$-module. Here the sum runs over all non-negative integers $a_1, \ldots, a_k$ such that $\sum_{i=1}^k i a_i = k$, and $S^a(M)$ means the symmetric tensor product of a $\mathbf{Z}$-module M such that $S^0(M) = \mathbf{Z}$.

We see that this is true for $k = 1$ and 2 from a general argument in group ring theory. For $k \geq 3$, however, it is still an open problem. In general, one of the most standard methods to study the augmentation quotients $Q^k(\mathrm{IA}_n)$ is to consider a natural surjective homomorphism π_k : $Q^k(\mathrm{IA}_n) \to Q^k(\mathrm{IA}_n^{\mathrm{ab}})$ induced from the abelianization $\mathrm{IA}_n \to \mathrm{IA}_n^{\mathrm{ab}}$ of IA_n. Furthermore, since $\mathrm{IA}_n^{\mathrm{ab}}$ is free abelian, we have a natural isomorphism $Q^k(\mathrm{IA}_n^{\mathrm{ab}}) \cong S^k(\mathcal{L}_{\mathrm{IA}_n}(1))$. Hence, in the conjecture above, we can detect $S^k(\mathcal{L}_{\mathrm{IA}_n}(1))$ in $Q^k(\mathrm{IA}_n)$ by the abelianization of IA_n.

Then we have a natural problem to consider: Determine the structure of the kernel of π_k. More precisely, clarify the $\mathrm{GL}(n, \mathbf{Z})$-module structure of $\mathrm{Ker}(\pi_k)$. In order to attack this problem, in this article we construct and study a certain homomorphism defined on $Q^k(\mathrm{IA}_n)$ whose restriction to $\mathrm{Ker}(\pi_k)$ is non-trivial. For a group G, let $\alpha_k = \alpha_{k,G} : \mathcal{L}_G(k) \to Q^k(G)$ be a homomorphism defined by $\sigma \mapsto \sigma - 1$. Then, we can construct a $\mathrm{GL}(n, \mathbf{Z})$-equivariant homomorphism

$$\mu_k : Q^k(\mathrm{IA}_n) \to \mathrm{Hom}_{\mathbf{Z}}(H, \alpha_{k+1}(\mathcal{L}_n(k+1)))$$

where $\mathcal{L}_n(k)$ is the k-th graded quotient of the lower central series of F_n. Furthermore, for the k-th Johnson homomorphism

$$\tau'_k : \mathcal{L}_{\mathrm{IA}_n}(k) \to \mathrm{Hom}_{\mathbf{Z}}(H, \mathcal{L}_n(k+1))$$

defined by $\sigma \mapsto (x \mapsto x^{-1}x^{\sigma})$, we show that $\mu_k \circ \alpha_k = \alpha^*_{k+1} \circ \tau'_k$ where α^*_{k+1} is a natural homomorphism induced from α_{k+1}. Since α_{k,F_n} is a $\mathrm{GL}(n, \mathbf{Z})$-equivariant injective homomorphism for each $k \geq 1$, if we identify $\mathcal{L}_n(k)$ with its image $\alpha_k(\mathcal{L}_n(k))$, we obtain $\mu_k \circ \alpha_k = \tau'_k$. Hence, the homomorphism μ_k can be considered as a lift of the Johnson homomorphism τ'_k. In the following, we naturally identify $\mathrm{Hom}_{\mathbf{Z}}(H, \mathcal{L}_n(k+1))$ with $H^* \otimes_{\mathbf{Z}} \mathcal{L}_n(k+1)$ for $H^* := \mathrm{Hom}_{\mathbf{Z}}(H, \mathbf{Z})$.

Historically, the study of the Johnson homomorphisms was originally begun in 1980 by D. Johnson [11] who determined the abelianization of the Torelli subgroup of the mapping class group of a surface in [12]. Now, there is a broad range of remarkable results for the Johnson homomorphisms of the mapping class group. (For example, see [10] and [15], [16], [17].) These works also inspired the study of the Johnson homomorphisms of $\mathrm{Aut}\, F_n$. Using it, we can investigate the graded quotients $\mathrm{gr}^k(\mathcal{A}_n)$ and $\mathcal{L}_{\mathrm{IA}_n}(k)$. Recently, good progress has been achieved through the works of many authors, for example, [6], [7], [9], [13], [15], [16], [17] and [19]. In particular, in our previous work [24], we determined the cokernel of the rational Johnson homomorphism $\tau'_{k,\mathbf{Q}} := \tau'_k \otimes \mathrm{id}_{\mathbf{Q}}$ for $2 \leq k \leq n-2$.

The main theorem of this article is

Theorem 1. *For $3 \leq k \leq n-2$, the $\mathrm{GL}(n,\mathbf{Z})$-equivariant homomorphism*

$$\mu_k \oplus \pi_k : Q^k(\mathrm{IA}_n) \to (H^* \otimes_{\mathbf{Z}} \alpha_{k+1}(\mathcal{L}_n(k+1))) \bigoplus Q^k(\mathrm{IA}_n^{\mathrm{ab}})$$

defined by $\sigma \mapsto (\mu_k(\sigma), \pi_k(\sigma))$ is surjective.

Next, we consider the framework above for a free metabelian group. Let $F_n^M := F_n/[[F_n,F_n],[F_n,F_n]]$ be a free metabelian group of rank n. By the same argument as the free group case, we can consider the IA-automorphism group IA_n^M and the Johnson homomorphism

$$\tau'_k : \mathcal{L}_{\mathrm{IA}_n^M}(k) \to H^* \otimes_{\mathbf{Z}} \mathcal{L}_n^M(k+1)$$

of $\mathrm{Aut}\, F_n^M$ where $\mathcal{L}_{\mathrm{IA}_n^M}(k)$ is the k-th graded quotient of the lower central series of IA_n^M, and $\mathcal{L}_n^M(k)$ is that of F_n^M. In our previous work [23], we studied the Johnson homomorphism of $\mathrm{Aut}\, F_n^M$, and determined its cokernel. In particular, we showed that there appears only the Morita obstruction $S^k H$ in $\mathrm{Coker}(\tau'_k)$ for any $k \geq 2$ and $n \geq 4$. We remark that in [23], we determined the cokernel of the Johnson homomorphism τ_k which is defined on the graded quotient of the Johnson filtration of $\mathrm{Aut}\, F_n^M$. Observing our proof, we verify that $\mathrm{Coker}(\tau'_k) = \mathrm{Coker}(\tau_k)$.

Now, similarly to the free group case, we can also construct a $\mathrm{GL}(n,\mathbf{Z})$-equivariant homomorphism

$$\mu_k : Q^k(\mathrm{IA}_n^M) \to \mathrm{Hom}_{\mathbf{Z}}(H, \alpha_{k+1}(\mathcal{L}_n^M(k+1)))$$

such that $\mu_k \circ \alpha_k = \alpha_{k+1}^* \circ \tau'_k$. Then we have

Theorem 2. *For $k \geq 2$ and $n \geq 4$, the $\mathrm{GL}(n,\mathbf{Z})$-equivariant homomorphism*

$$\mu_k \oplus \pi_k : Q^k(\mathrm{IA}_n^M) \to (H^* \otimes_{\mathbf{Z}} \alpha_{k+1}(\mathcal{L}_n^M(k+1))) \bigoplus S^k((\mathrm{IA}_n^M)^{\mathrm{ab}})$$

defined by $\sigma \mapsto (\mu_k(\sigma), \pi_k(\sigma))$ is surjective.

Acknowledgments

This research is supported by a JSPS Research Fellowship for Young Scientists and the Global COE program at Kyoto University.

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